DIAS **Delhi Institute for Administrative Services**
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ALL INDIA TEST SERIES CSE-2023

Candidate 's Information

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4. **SUBJECT:-** Electricity and magnetism 8-TS
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Excellent

FOR OFFICE USE ONLY:-

Q.NO	MARKS
1.	35
2.	
3.	28
4.	32
5.	33½
6.	33
7.	
8.	

TOTAL MARKS	162/250

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AITS 2023: Test 7 (E&M)

UPSC

Answer Questions in NOT MORE THAN the Word Limit specified for each in the Parenthesis.
(Specimen Answer Booklet - For Practice Purpose Only)

Q1(a) Gauss's theorem states that flux of electric field through closed surface is $1/\varepsilon_0$ times charge enclosed.

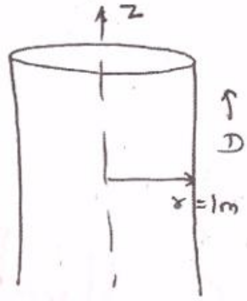

Fig 1. Given cylinder

$\Rightarrow \vec{\nabla}\cdot\vec{D} = \rho$. As $\vec{D}$ is only in $\hat{z}$, in cylindrical coordinates, we have,

$$\frac{\partial}{\partial z}(\vec{D}) = \rho = r\cos^2\phi$$

$$\Rightarrow \rho_{(1,\pi/4,3)} = \boxed{0.5\ c/m^3}$$

> Total charge enclosed = $\int \rho\, d\tau$

$$\Rightarrow Q_{enc} = \int r\cos^2\phi \cdot r \cdot d\phi \cdot dr\, dz$$

$$\Rightarrow Q_{enc} = \int_0^1 r^2 dr \int_0^{2\pi} \cos^2\phi\, d\phi \int_{-2}^{2} dz$$

$$\Rightarrow Q_{enc} = \frac{1}{3} \times \pi \times 4 \Rightarrow \boxed{Q_{enc} = 4\pi/3\ C}$$

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Hence the charge density at $(1, \pi/4, 3)$ is $\boxed{0.5\ c/m^3}$ while charge enclosed in cylinder is $\boxed{\sim 4.18\ C}$.

Q1 (b)

Gauss' theorem gives the electric field ($\vec{E}$) is charge distribution is known.

We know,

$$\oint \vec{E}\cdot\vec{ds} = Q_{enc}/\epsilon_0$$

$$\Rightarrow \vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 .$$

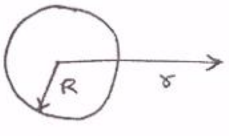

Fig 1. Given distribution

In spherical coordinates,

$\vec{\nabla}\cdot\vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\vec{E}\right)$ as we have θ, ϕ symmetry

(A) For $r < R$:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E\right) = \frac{\rho_0 r}{R} \Rightarrow r^2\vec{E} = \int_0^r \frac{\rho_0 r^3}{R}\,dr$$

$$\Rightarrow \boxed{E_{in} = \frac{\rho_0 r^2}{4R\epsilon_0}\hat{r}}$$

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(B) For $r > R$:

Total charge enclosed $= Q_{enc} = \int \rho\, d\tau$

$$\Rightarrow Q_{enc} = \int_0^R \frac{\rho_0 r}{R}\cdot 4\pi r^2 dr = \pi\rho_0 R^3$$

$$\Rightarrow |\vec{E}|\cdot 4\pi r^2 = \frac{\pi\rho_0 R^3}{\epsilon_0} \Rightarrow \boxed{E_{out} = \frac{\rho_0 R^3}{4\epsilon_0 r^2}\hat{r}}.$$

Electric field can further be used to estimate potential.

Q1 (c)

Properties of EM wave like velocity, characteristic impedance, etc. can provide field parameters like $\vec{H}$ & $\vec{E}$

(A) ε_r :

We know $Z = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_r}{\varepsilon_r}} \cdot Z_0 \Rightarrow \varepsilon_r = \left(\frac{Z_0}{Z}\right)^2 \mu_r$.

where $Z_0 = 377\,\Omega \Rightarrow \boxed{\varepsilon_r = 4}$

(B) ω :

We know, $\vec{B} = \frac{\vec{k} \times \vec{E}}{\omega} \Rightarrow H = \frac{|k||E| \sin\theta}{\mu_0 \mu_r \omega}$ } As $E \perp k$, $\theta = 90°$

$\Rightarrow \omega = \frac{E}{H}\left(\frac{1}{\mu}\right) = \frac{Z}{\mu} = \boxed{1.5 \times 10^8 \text{ rad/s}}$

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(C) $\vec{E}$:

We know from Maxwell's equations, $\vec{E} = \frac{\vec{H} \times \vec{k}}{\varepsilon \cdot \omega}$

Here $k = 1\hat{z}$

$$\Rightarrow \vec{H} \times \vec{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -0.1\cos(\omega t - z) & 0.5\sin(\omega t - z) & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{1}{\varepsilon\omega}\left[0.5\sin(\omega t - z)\hat{i} + 0.1\cos(\omega t - z)\hat{j}\right]$$

Substituting ε, ω,

$$\boxed{\vec{E} = 94.12\sin(\omega t - z)\, a_x + 18.8\cos(\omega t - z)\, a_y .}$$

$\vec{E}, \vec{H}$ can b/e further used to calculate Poynting Vector.

Q1 (d)

Series RL circuit can be used as integrator & differentiator based on operating frequency.

(A) Integrator :

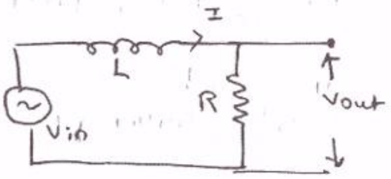

Fig 1. Integrator Circuit

> In the circuit shown,

$V_o = IR$.

We know, $V_L = L\frac{dI}{dt}$

$\Rightarrow I = \frac{1}{L}\int V_L dt$. Now if V_L is made such that $V_L \approx V_{in}$,

$V_o = \frac{R}{L}\int v_{in} dt$. For $V_L \approx V_{in}$, $\omega L >>> R$.

$\Rightarrow \boxed{\omega >> R/L}$.

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(B) Differentiator :

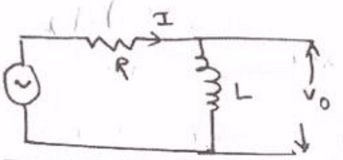

Fig 2. Differentiator Circuit

> $V_o = L\frac{dI}{dt}$. Now $V_R = IR$

$\Rightarrow R\frac{dI}{dt} = \frac{dV_R}{dt}$. If $V_R \approx V_{in}$,

$V_o = \frac{L}{R}\cdot\frac{dV_{in}}{dt}$. For $V_R \approx V_{in}$,

$IR >> I(\omega L)$

$\Rightarrow \boxed{\omega << R/L}$.

However, these circuits have impedance. Hence Op-Amp are used in circuits.

Q1(c). Planck's radiation formula successfully explained the black-body radiation curve given by Lummer & Pringsheim.

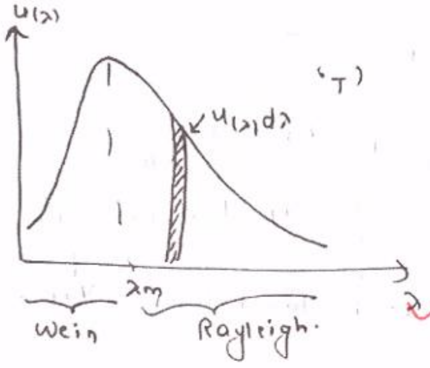

Fig 1. Spectral radiation of black body

Planck's law states that:

$$u(\nu) = \frac{8\pi\nu^2}{c^3}\frac{h\nu}{e^{h\nu/kT}-1}$$

where $u(\nu)$ = energy radiated by black-body between ν & $d\nu$. In terms of λ, $u(\lambda) = \frac{8\pi hc}{\lambda^5}\frac{d\lambda}{e^{hc/\lambda kT}-1}$

① At low wavelengths,

$$e^{hc/\lambda kT} >> 1 \Rightarrow u(\lambda) = \frac{8\pi hc}{\lambda^5}\frac{d\lambda}{e^{hc/\lambda kT}} = \frac{f(\lambda)d\lambda}{\lambda^5}$$

This is Wein's distribution law

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② At high wavelengths,

$$e^{hc/\lambda kT} < 1 \approx 1 + hc/\lambda kT$$

$$\Rightarrow u(\lambda) = \frac{8\pi hc}{\lambda^5}\cdot\frac{d\lambda}{hc/\lambda kT} = \frac{8\pi}{\lambda^4}(kT)\,d\lambda \Big\} \text{Rayleigh law}$$

Hence, Planck was able to explain both ends of spectrum in black body radiation

Q3 (a)

Molecular polarizability is the extent of dipole moment induced in a dielectric in response to an external electric field.

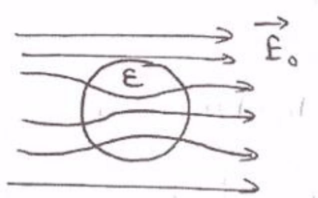

Fig 1. Dielectric in external field

$\Rightarrow \vec{p}_m = \alpha \vec{E}_m$ where

$\vec{p}_m$ - induced dipole moment,

α = molecular polarizability

Now $\vec{E}_m = \vec{E}_0 + \vec{E}_p$.

We know $\vec{E}_p = \frac{-\vec{P}}{3\varepsilon_0}$. Also, $\vec{P} = \frac{\vec{p}}{d\tau}$ where

$\vec{P}$ = Polarization. Assuming n dipoles per unit volume,

$$\frac{\vec{P}}{n} = \alpha\left[\varepsilon_0 - \frac{\vec{P}}{3\varepsilon_0}\right] \Rightarrow \vec{P}\left[\frac{1}{n\alpha} + \frac{1}{3\varepsilon_0}\right] = \vec{E}_0$$

We know $\vec{P} = \varepsilon_0 (k-1)\vec{E}$ } k = dielectric constant.

$$\Rightarrow \varepsilon_0 (k-1)\left[\frac{1}{n\alpha} + \frac{1}{3\varepsilon_0}\right] = 1$$

$$\Rightarrow \frac{\varepsilon_0}{n\alpha} + \frac{1}{3} = \frac{1}{k-1}$$

$$\Rightarrow \boxed{\frac{k+2}{k-1} = \frac{3\varepsilon_0}{n\alpha}}$$

This is known as Clausius-Mosotti equation

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It provides extent of polarizability if dielectric constant is known.

Limitations of Clausius Mosotti

① Valid only for ~~h~~ isotropic dielectric.

② Valid only for linear dependence of $\vec{P}$ on $\vec{E}$

Q3 (b)

Potentials are mathematical constructs which provide the field parameters like $\vec{B}$.

We know, $\vec{\nabla} \times \vec{A} = \vec{B}$.

$$\Rightarrow \vec{B} = \begin{vmatrix} a_x & a_y & a_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 y & y^2 x & -4xyz \end{vmatrix}$$

$$\Rightarrow \vec{B} = [-4xz]a_x + [4yz]a_y + [y^2 - x^2]a_z .$$

$$\Rightarrow \vec{B}_{(-1,2,5)} = [-4(-1)(5)]a_x + [4(2)(5)]a_y + [2^2 - 1^2]a_z$$

$$\Rightarrow \boxed{\vec{B}_{(-1,2,5)} = 20a_x + 40a_y + 3a_z}$$

> Flux of magnetic field through a surface is given by $\phi = \int \vec{B} \cdot \vec{ds}$

Where ϕ = Flux = # magnetic field lines cross the surface per unit time.

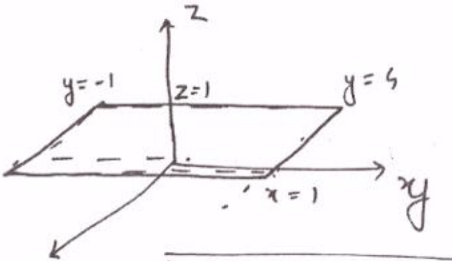

Fig 1 - Given surface

In this case, $\vec{ds} = dx\,dy\,\hat{a}_z$ as surface is in x-y plane

$$\Rightarrow \phi = \int_{ds} (y^2 - x^2)\cdot dx\,dy$$

$$\Rightarrow \phi = \int_{x=0}^{1}\int_{y=-1}^{4} y^2\,dx\,dy - \int_{x=0}^{1}\int_{y=-1}^{4} x^2\,dx\,dy$$

$$\Rightarrow \phi = \frac{1}{3}\left[4^3 - (-1)^3\right] - \frac{1}{3}\left(4-(-1)\right)]$$

$$\Rightarrow \phi = \frac{1}{3}\times 65 - \frac{5}{3} = \boxed{20\ Wb}$$

$\frac{10}{15}$

Potentials can further be used to estimate current density, electric field, etc.

Q.3 (c)

A plane wave propogating in a conducting medium gives attenuated wave equation leading to exponentially declining electric $(\vec{E})$ & magnetic $(\vec{B})$ fields

From Maxwell's equations, for conductor $(\rho = 0)$

① $\vec{\nabla} \cdot \vec{D} = 0$ ② $\vec{\nabla} \cdot \vec{B} = 0$

③ $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ ④ $\vec{\nabla} \times \vec{H} = \vec{J} + \vec{J_D}$.

From eq. ③ & ④,

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B}) \Rightarrow -\nabla^2 \vec{E} = -\mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

As $\vec{J} = \sigma \vec{E}$, $J_D = \varepsilon \partial \vec{E} / \partial t$.

$$\Rightarrow \nabla^2 \vec{E} - \mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \quad \Big\} \text{ Attenuated wave eq.}$$

Let $\vec{E} = E_0 e^{j(\vec{k} \cdot \vec{r} - \omega t)}$ where $k = \beta + j\alpha$.

Substituting,

$$-k^2 + j\mu\sigma\omega + \mu\varepsilon\omega^2 = 0.$$

$$\Rightarrow \beta^2 - \alpha^2 + 2j\alpha\beta = j\mu\sigma\omega + \mu\varepsilon\omega^2.$$

$$\Rightarrow \beta^2 - \alpha^2 = \mu\varepsilon\omega^2 \quad — ①$$

$$2\alpha\beta = \mu\sigma\omega \quad — ②$$

On solving, we get,

$$\beta = \sqrt{\frac{\mu\varepsilon_0\omega^2}{2}}\left[1+\sqrt{1+\left(\frac{\sigma}{\varepsilon_0\omega}\right)^2}\right]^{1/2}$$

Given $\sigma = 3$ mhos/m, $\omega = 10^8$.

$$\Rightarrow \frac{\sigma}{\varepsilon_0\omega} = 3.38\times10^3 \text{. As } \frac{\sigma}{\varepsilon_0\omega} >> 1$$

$$\Rightarrow \beta \approx \sqrt{\frac{\mu\sigma\omega}{2}} = \boxed{61.4\ m^{-1}}$$

not in

$\vec{E}$ & $\vec{H}$ are

From eq. ②, $\alpha = \frac{\mu\sigma\omega}{2\beta} = \sqrt{\frac{\mu\sigma\omega}{2}}$.

$$\Rightarrow \boxed{\alpha = \beta = 61.4 m^{-1}}$$

In Conducting media

same phase.

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We know, $\vec{B} = \frac{\vec{k}\times\vec{E}}{\omega} \Rightarrow \vec{H} = \frac{\vec{k}\times\vec{E}}{\mu\omega}$. $\}\ \vec{k} = \beta\hat{z}$

$$\Rightarrow \vec{H} = \frac{1}{\mu\omega}\left[61.4\hat{z} \times 2e^{-\alpha z}\sin(10^8 t - \beta z)\,a_y\right]$$

$$\Rightarrow \boxed{\vec{H} = 4.8\times10^{-2}\, e^{-\alpha z}\sin(10^8 t - \beta z)(-a_x)\ A/m}$$

$-\pi/4$

$H_0 = \frac{E_0}{\eta}$

$\eta = \sqrt{\frac{\mu\omega}{\sigma}}$

$H_0 = 69.1$

Where α & β are as calculated.

This verifies that electric & magnetic field in EM wave are in same phase.

Tan

$\tan 2\theta = \frac{\sigma}{\omega\varepsilon} \Rightarrow \theta = \pi/4$

Q4 (a).

Electromagnetic field tensor is 4×4 anti-Symmetric matrix of rank 2 which gives all components of the field parameters $\vec{B}$ & $\vec{E}$.

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}$$ } where $F_{\mu\nu}$ is element of EM tensor.

Defining $\vec{B} = \vec{\nabla} \times \vec{A}$ & $\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$,

Under Lorentz Gauge condition, we get,

$$\left.\begin{aligned} \Box^2 \vec{A} &= -\mu_0 \vec{J} \\ \Box^2 \phi &= -\rho/\varepsilon_0 \end{aligned}\right\}$$

Taking components,

$$F_{\mu\nu} = \begin{bmatrix} 0 & B_3 & -B_2 & -jE_1/c \\ -B_3 & 0 & B_1 & -jE_2/c \\ B_2 & -B_1 & 0 & -jE_3/c \\ jE_1/c & \frac{jE_2}{c} & \frac{jE_3}{c} & 0 \end{bmatrix}.$$

Invariance of Maxwell's equation

> Invariance refers to retaining of physical form by laws of physics across inertial frames.

> We define 4-vector $J_\mu = (\vec{J}, ic\rho)$

From Lorentz gauge condition,

$\square^2 \vec{A} = -\mu_0 \vec{J}$ & $\square^2 \phi = -\rho/\varepsilon_0$

$\Rightarrow \square^2 (i\phi/c) = (ic\rho)\mu_0 = -\mu_0 J_4$.

$\Rightarrow \square^2 A_\mu = -\mu_0 J_\mu$ where $A_\mu = (\vec{A}, i\phi/c)$.

To prove Maxwell eq. are invariant,

$\square'^2 A'_\mu = -\mu_0 J'_\mu$. We know $\square'^2 = \square^2$.

As J_μ is a 4-vector, it can be transformed using transformation matrix.

i.e. $J'_\nu = \alpha_{\mu\nu} J_\mu$. & $\alpha_{\mu\nu}$ = transformation matrix.

$$\Rightarrow \square^2 \begin{bmatrix} A_1' \\ A_2' \\ A_3' \\ A_4' \end{bmatrix} = \square^2 \begin{bmatrix} \alpha & 0 & 0 & i\alpha\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\alpha\beta & 0 & 0 & \alpha \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix}$$

$\Rightarrow \square^2 A_1' = \square^2 [\alpha A_1 + i\alpha\beta A_4] = -\mu_0 J_1'$

$\square^2 A_2' = \square^2 A_2 = -\mu_0 J_2 = -\mu_0 J_2'$

$\square^2 A_3' = \square^2 A_3 = -\mu_0 J_3 = -\mu_0 J_3'$

$\square^2 A_4' = \square^2 (-i\alpha\beta A_1 + \alpha A_4) = -i\alpha\beta(-\mu_0 J_1) + \alpha(\mu_0 J_4)$

$= -\mu_0 J_4'$

$$\Rightarrow \boxed{\Box^2 A_\mu' = -\mu_0 J_\mu'}$$

Hence, Maxwell's equations are invariant to Lorentz transformations.

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Q4(b) . Poynting vector ($\vec{S}$) gives the intensity i.e (energy - per unit area - time) carried by an electromagnetic wave out of a bounded surface

We know $\vec{S} = \vec{E} \times \vec{H}$.

$$\Rightarrow \vec{\nabla} \cdot \vec{S} = \vec{\nabla} \cdot (\vec{E} \times \vec{H}) = (\vec{\nabla} \times \vec{E}) \cdot \vec{H} - (\vec{\nabla} \times \vec{H}) \cdot \vec{E} .$$

From Maxwell's equations, we know,

① $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ ② $\vec{\nabla} \times \vec{H} = \vec{J} + \vec{J}_D$

$$\Rightarrow \vec{\nabla} \cdot \vec{S} = -\frac{\partial \vec{B}}{\partial t} \cdot \vec{H} - \vec{J} \cdot \vec{E} - \frac{\partial \vec{D}}{\partial t} \cdot \vec{E} .$$

$$\Rightarrow -\frac{\partial}{\partial t} \left(\frac{\vec{B} \cdot \vec{H} + \vec{D} \cdot \vec{E}}{2} \right) = \vec{\nabla} \cdot \vec{S} + \vec{J} \cdot \vec{E} .$$

$$\Rightarrow \int \vec{\nabla} \cdot \vec{S} \cdot d\tau + \int \vec{J} \cdot \vec{E} \cdot d\tau = \int -\frac{\partial U}{\partial t} \, d\tau .$$

where U = energy density = $\frac{1}{2} [\vec{B} \cdot \vec{H} + \vec{D} \cdot \vec{E}]$

$$\Rightarrow \boxed{\underbrace{\oint_{da} \vec{S} \cdot \vec{da}}_{c} + \underbrace{\int_{dv} \vec{J} \cdot \vec{E} \, d\tau}_{b} = \underbrace{\int_{dv} -\frac{\partial U}{\partial t} \, d\tau}_{a}}$$

⇒ Poynting theorem gives the energy conservation theorem in electromagnetism which states:

Rate of change of energy density of em wave (a) = Energy carried by em wave out of a surface (c) + Work done on field charge (b)

In case of non-conducting medium, $\vec{J} = 0$

⇒ Poynting theorem becomes $\vec{\nabla}\cdot\vec{S} + \frac{\partial U}{\partial t} = 0$.

Poynting theorem is used to estimate the energy required to transmit em waves in satellite communication, mobile networks, etc.

$\frac{9\frac{1}{2}}{15}$

Q.4(c)(i) . An A.C circuit is powered by a sinusoidal voltage with lead/lag in current.

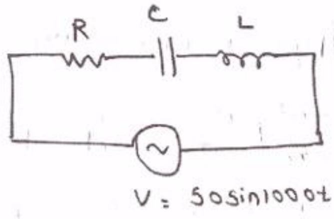

Fig 1. RLC series circuit

We know, $P = VI$.

Let $V = V_0 \sin \omega t$,

$I = I_0 \sin(\omega t + \phi)$

$\Rightarrow P = V_0 I_0 \sin(\omega t) \sin(\omega t + \phi)$

Taking time average,

$\langle P \rangle = \frac{V_0 I_0}{2} \cos\phi$ where ϕ = phase difference

$\Rightarrow \langle P \rangle = \frac{V_0^2 R}{2Z^2}$ as $\cos\phi = \frac{R}{Z}$, $I_0 = \frac{V_0}{Z}$.

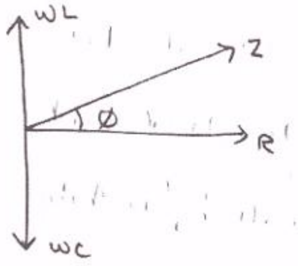

Fig 2. Phasor diagram

Here $Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$

$\Rightarrow Z = \sqrt{100^2 + \left(5000 - \frac{1}{2\times10^{-4}}\right)}$

$\Rightarrow Z = 100\,\Omega$

$\Rightarrow P = \frac{V_0^2}{2R} = \boxed{12.5\,W}$

Quality factor Q : Gives the sharpness of resonance & is defined as $Q = \frac{\text{Resonant } \omega}{\text{Band-width}}$.

For series LCR circuit,

$$Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 R C} \implies \boxed{Q = 50}.$$

where $\omega_0 = \frac{1}{\sqrt{LC}} = 1000$ rad/sec.

$$\implies Q = \frac{1000}{BW} \implies \boxed{\text{Band width} = 20}.$$

Hence, in given circuit, average power consumed $\boxed{\langle P \rangle = 12.5\,W}$, $\boxed{Q = 50}$ & $\boxed{BW = 20}$.

Lower the bandwidth, sharper is the resonance ⟹ Better is the synchronisation between source & system.

$9\frac{1}{2}$ / 15

Q.5 (a)

Gauge transformation refers to change of inertial frame i.e. change of scale (gauge). By choosing appropriate gauge, equations can be simplified & field parameters $\vec{B}$ & $\vec{E}$ obtained more easily.

Lorentz gauge condition is $\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t}$.

We define 4-vector A_μ as $(\vec{A}, i\phi/c)$

We know, $x_\mu = (\vec{x}, ict)$

$$\Rightarrow \vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = \vec{\nabla}\cdot\vec{A} + \frac{\partial(i\phi/c)}{\partial(ict)} = \Box\cdot A_\mu$$

$\Rightarrow$ Lorentz gauge conditions becomes: $\sum_{i=1}^{4} \frac{dA_i}{dx_i}$. We know,

$$\Box' = \Box \Rightarrow \Box' A_\mu' = \Box\cdot A_\mu' = \text{Invariant}$$

As A_μ is a 4-vector whose norm is invariant.

Hence, Lorentz gauge condition is Lorentz invariant

$6\frac{1}{2}/10$

Q5 (b)

Property of EM wave that E, B, k form triad gives $\vec{B}$ when $\vec{E}$ is known.

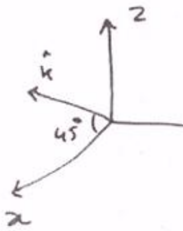

Fig1. EM wave

> Given $\vec{k} = \frac{1}{\sqrt{2}}(\hat{i} + \hat{k})$

We know, $\vec{B} = \frac{\vec{k} \times \vec{E}}{\omega}$.

$$\Rightarrow \vec{B} = \frac{1}{\omega}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -E_0 \sin\omega t & 2E_0 \cos\omega t & E_0 \sin(\omega t) \end{vmatrix}$$

$$\Rightarrow \vec{B} = \frac{1}{\omega}\left[-\sqrt{2}E_0\cos\omega t\,\hat{i} - \sqrt{2}\sin\omega t(E_0)\hat{j} + \sqrt{2}E_0\cos\omega t\,\hat{k}\right]$$

$$\Rightarrow \boxed{\vec{B} = \frac{\sqrt{2}E_0}{\omega}\left[-\cos\omega t\,\hat{i} - \sin\omega t\,\hat{j} + \cos\omega t\,\hat{k}\right]}$$

$$\Rightarrow \vec{H} = \frac{\vec{B}}{\mu_0} = \frac{\sqrt{2}E_0}{\mu_0\omega} \quad \Big\} \quad \vec{S} = \vec{E} \times \vec{H} = \text{Poynting vector}$$

$$\Rightarrow \vec{S} = \frac{\sqrt{2}E_0^2}{\mu_0\omega}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin\omega t & 2\cos\omega t & \sin\omega t \\ -\cos\omega t & -\sin\omega t & \cos\omega t \end{vmatrix}$$

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$$\Rightarrow \boxed{\vec{S} = \frac{\sqrt{2}E_0^2}{\mu_0\omega}\left[(2\cos^2\omega t + \sin^2\omega t)\hat{i} + (2\cos^2\omega t + \sin^2\omega t)\hat{k}\right]}$$

$$= \frac{\sqrt{2}E_0^2}{\mu_0\omega}(1+\cos^2\omega t)(\hat{i}+\hat{k})$$

This shows Poynting vector is energy carried by wave in propogation direction

Q.5(c) Dipole is a pair of equal & opposite charge separated by finite distance.

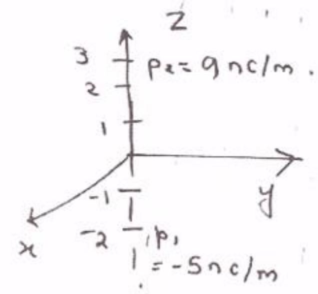

Fig1. Dipole placement

> We know, potential due to dipole = $\frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3}$

$r_1 = 2\hat{k}$; $r_2 = -3\hat{k}$

$$\Rightarrow V = \frac{1}{4\pi\varepsilon_0}\left[\frac{-9}{3^2} + \frac{-5}{2^2}\right] nC$$

$$\Rightarrow \boxed{V = -20.22\,V}$$

∴ Potential at origin due ~~to~~ given 2 dipoles is $\boxed{-20.2V}$.

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This can be further utilized to estimate electric field at origin.

Q5 (d)

Displacement current is an imaginary current which accounts for varying electric field making total current solenoidal.

Displacement current $I_D = A\frac{\partial \vec{D}}{\partial t} = A\varepsilon\frac{\partial \vec{E}}{\partial t}$.

For a parallel plate capacitor, we know, $\vec{E} = \frac{V}{d}$

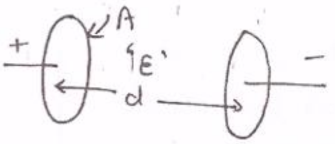

Fig 1. Parallel plate capacitor

where

$\Rightarrow \frac{\partial \vec{E}}{\partial t} = \frac{1}{d} \cdot \frac{dV}{dt} = \frac{1}{d}\frac{dV}{dt}$.

$\Rightarrow I_D = \frac{A\varepsilon}{d} \cdot \frac{dV}{dt}$.

$\Rightarrow I_D = \frac{A\varepsilon}{d}\frac{d}{dt}(50\sin 10^3 t) = \frac{50A\varepsilon \times 10^3}{d} \cos 10^3 t$.

Substituting values, $\boxed{I_D = 1.47\times 10^{-7} \cos 10^3 t \text{ A}}$

As $I_D << I$, it is generally not observed

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Q.5 (c)

Vector potential $(\vec{A})$ is a mathematical construct which gives $\vec{B}$ (magnetic field intensity)

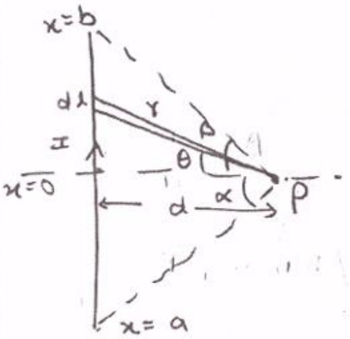

Fig 1. Wire carrying current

We know, vector potential,

$\vec{A} = \frac{\mu_0 I}{4\pi}\int \frac{\vec{dl}}{r}$. Consider element dl at angle θ from P.

$\Rightarrow r = \frac{d}{\cos\theta}$; $l = d\tan\theta$
$\Rightarrow dl = d\sec^2\theta\, d\theta$

$$\Rightarrow \vec{A} = \frac{\mu_0 I}{4\pi}\int_{-\alpha}^{\beta} \frac{d\sec^2\theta\, d\theta}{d/\cos\theta}$$

$$\Rightarrow \vec{A} = \frac{\mu_0 I}{4\pi}\int_{-\alpha}^{\beta} \sec\theta\, d\theta = \frac{\mu_0 I}{4\pi}\Big[\ln(\sec\theta + \tan\theta)\Big]_{-\alpha}^{\beta}$$

$$\Rightarrow \vec{A} = \frac{\mu_0 I}{4\pi} \ln \frac{(\sec\beta + \tan\beta)}{(\sec\alpha - \tan\alpha)}$$ Substituting α & β,

$$\boxed{\vec{A} = \frac{\mu_0 I}{4\pi} \ln \frac{\left(\frac{\sqrt{d^2+b^2}}{d} + \left(\frac{b}{d}\right)\right)}{\left(\sqrt{\frac{(d^2+a^2)}{d}} - \left(\frac{a}{d}\right)\right)}}$$

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This can be further used to estimate $\vec{B}$ as $\underline{\vec{\nabla}\times\vec{A}}$).

Q. 6
(a)

Capacitor discharging through inductor & resistor works like tank-circuit where energy is shuttled between inductor & capacitor.

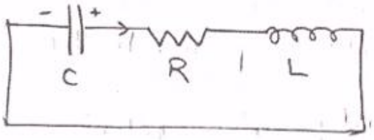


Fig 1. Capacitor discharge

From Kirchoff's law,

$$V_C + V_R + V_L = 0$$

$$\Rightarrow \frac{Q}{C} + IR + L\frac{dI}{dt} = 0.$$

$$\Rightarrow L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}\frac{dQ}{dt} = 0$$

$$\Rightarrow \frac{d^2Q}{dt^2} + 2c\frac{dQ}{dt} + \omega_0^2 \ldots = 0 \quad \left.\begin{matrix} 2c = R/L \\ \omega_0^2 = 1/LC. \end{matrix}\right\}$$

Let the solution be $Q = Q_0 e^{\alpha t}$.

$\Rightarrow [\alpha^2 + 2\alpha c + \omega_0^2] = 0$. We get 3 cases:

Case-I: $c > \omega_0 \Rightarrow R > R_0$ [over-damped]

$$Q = Q_0 e^{-ct}\left[c_1 e^{\sqrt{c^2-\omega_0^2}\,t} + c_2 e^{-\sqrt{c^2-\omega_0^2}\,t}\right]$$

i.e. it is exponentially dying oscillation.

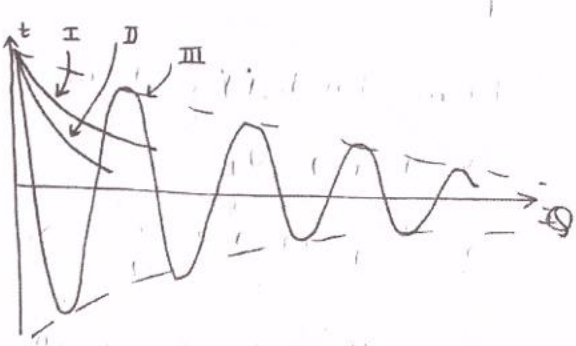
Case-II: $c = \omega_0 \Rightarrow R = R_0$ [critically damped]

$$\Rightarrow Q = Q_0 [c_1 + c_2 t] e^{-\frac{R}{2L}\cdot t}$$

This is also exponentially dying oscillation.

Case-III : $c < \omega_0 \Rightarrow R < R_0$ (under-damped case).

$$\Rightarrow Q = Q_0 e^{-R/2L\,t} \sin\left[\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\,.\,t + \phi\right].$$



Here, current falls off exponentially in sinusoidal pattern.

Fig 2. 3 cases of current variation

Let $\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = \omega$

$$\Rightarrow Q = Q_0 e^{-ct} \sin(\omega t + \phi)$$

Total energy $= \frac{1}{2}\frac{Q^2}{c} + \frac{1}{2} L I^2$, $I = \frac{dQ}{dt}$

$$\Rightarrow E = \frac{Q_0^2}{2c} e^{-2ct} \sin^2(\omega t + \phi) + \frac{LQ_0^2\omega^2}{2} \cos^2(\omega t + \phi))$$

We know, $\omega^2 = 1/LC$

$$\Rightarrow E_{total} = \frac{Q_0^2}{2c} e^{-2ct} = \boxed{E_{max}\, e^{-2ct}}$$

Quality factor is defined as ratio of energy stored to energy lost per cycle

$$\Rightarrow Q = 2\pi \left(\frac{E_{stored}}{E_{lost}/cycle}\right)$$

$$\frac{dE}{dt} = 2c \cdot E_{max} e^{-2ct}.$$

$$\Rightarrow Q = 2\pi \left[\frac{E_{max} e^{-2ct}}{2c \cdot E_{max} e^{-2ct} (T)} \right] = \boxed{\frac{\omega_0}{2c}}.$$

> In this case, it represents, sharpness of resonance i.e. bandwidth.

$$\text{As } Q = \frac{1}{\sqrt{LC} \times 2 \times \frac{R}{2L}} = \boxed{\frac{1}{R}\sqrt{\frac{L}{C}}} = \frac{\omega_0}{B\omega},$$

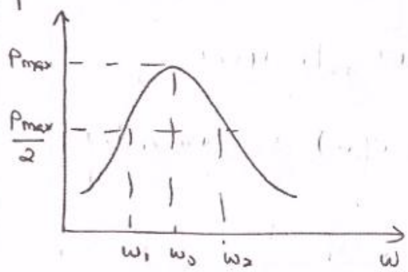


Fig 3. Resonance

If Q is high, $\omega_1 \sim \omega_2$ is very low ⇒ resonance is sharper..

Q can be increased by decreasing R or C.

Tank-circuits are used in oscillators to produce AC from DC.

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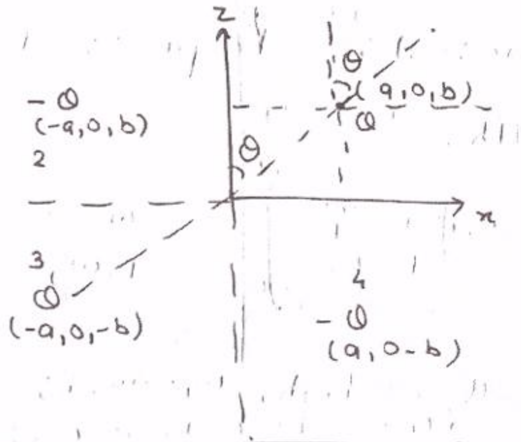
Q 6 (b)

Lord kelvin theorem (Uniqueness theorem) states for given boundary condition, there exists only one & one solution to laplace.



> As V = 0 on the given plates, by method of images, we can have charges as shown in figure.

Fig 1. Method of images

∴ V(x,y,z) is given by :-

$$V_{(x,y,z)} = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{\sqrt{(x-a)^2+y^2+(z-b)^2}} + \frac{Q}{\sqrt{(x+a)^2+y^2+(z+b)^2}}\right]$$

$$-\frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{\sqrt{(x+a)^2+y^2+(z-b)^2}} + \frac{Q}{\sqrt{(x-a)^2+y^2+(z+b)^2}}\right]$$

> Force on charge Q due to plates = Force on charge Q due to image charges.

$$\Rightarrow F_{12} = \frac{1}{4\pi\varepsilon_0}\,\frac{-Q^2}{(2a)^2}\,\hat{i}$$

$$F_{14} = \frac{1}{4\pi\varepsilon_0}\,\frac{-Q^2}{(2b)^2}\,\hat{k}$$

$$F_{13} = \frac{1}{4\pi\varepsilon_0}\,\frac{Q^2}{(2a)^2+(2b)^2}.$$ Taking components,

$$F_{13} = \frac{1}{4\pi\varepsilon_0}\,\frac{Q^2}{4(a^2+b^2)}\cdot\left[\sin\theta\,\hat{i}+\cos\theta\,\hat{k}\right]$$

$$\Rightarrow F_{13} = \frac{1}{4\pi\varepsilon_0}\,\frac{Q^2}{4(a^2+b^2)}\times\frac{1}{\sqrt{a^2+b^2}}\left[a\hat{i}+b\hat{k}\right]$$

2

$$\Rightarrow F_{total} = \frac{Q^2}{16\pi\varepsilon_0}\left[\frac{a}{(\sqrt{a^2+b^2})^3}-\frac{1}{a^2}\right]\hat{i}$$

$$+\frac{Q^2}{16\pi\varepsilon_0}\left[\frac{b}{(a^2+b^2)^{3/2}}-\frac{1}{b^2}\right]\hat{k}$$

Method of images can further be used to estimate induced surface charge density.

$$\frac{9\frac{1}{2}}{15}$$

Q.6 (c)

A plane, electromagnetic wave incident on a boundary undergoes reflection & transmission.

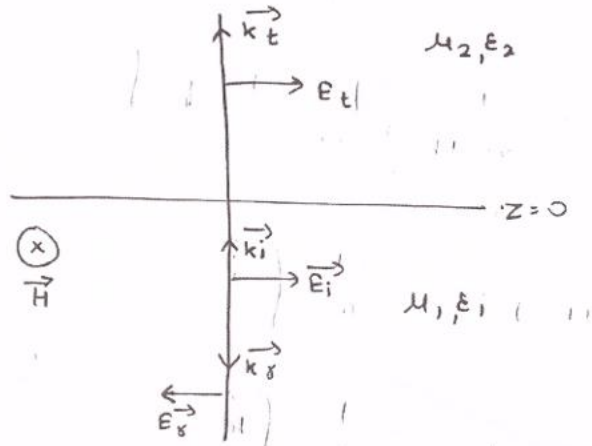


Fig 1. Incident wave with $E_{\parallel}$ to plane of incidence

> Consider a wave incident from $z<0$.

> Boundary conditions are :-

① $E_{\parallel(\text{above})} = E_{\parallel(\text{below})}$

② $H_{\parallel(\text{above})} = H_{\parallel(\text{below})}$

From the boundary conditions,

$H_i + H_r = H_t$ (whole H is ∥ to surface)

$E_i - E_r = E_t$ (As it is normal incidence)

We know, characteristic impedance, $Z = \frac{E}{H}$

$\Rightarrow H = \frac{E}{Z} = \frac{nE}{Z_0 \mu_r}$ } n = refractive index, μ_r = relative permeability

Assuming $\mu_1 = \mu_2 = 1$,

$E_i - E_r = E_t$ — ①

$n_1 E_i + n_1 E_r = n_2 E_t$ — ②

$$\Rightarrow \frac{E_t}{E_i} = t = \frac{2n_1}{n_1+n_2}$$. Substituting in eq. ①,

$$\frac{E_r}{E_i} = \left(\frac{n_1-n_2}{n_1+n_2}\right) = r$$; We know intensity $S \propto E^2$

$$\Rightarrow \frac{S_r}{S_i} = \left(\frac{E_r}{E_i}\right)^2 = \boxed{\left(\frac{n_1-n_2}{n_1+n_2}\right)^2} = R$$

$$\frac{S_t}{S_i} = \frac{n_2}{n_1}\left(\frac{E_t}{E_i}\right)^2 = \boxed{\frac{4n_1n_2}{(n_1+n_2)^2}} = T .$$

As can be seen, $R+T=1$

⇒ Proves conservation of energy i.e. intensity of reflected + intensity of transmitted wave = intensity of incident wave

Here, R, T, r, t are called Fresnel coefficient.

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